

Submean Variance Bound for Effective Resistance of Random Electric Networks

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Abstract: We study a model of random electric networks with Bernoulli resistances. In the case of the lattice \mathbb{Z}^2 , we show that the point-to-point effective resistance between 0 and a vertex v has a variance of order at most $(\log |v|)^{\frac{2}{3}}$, whereas its expected value is of order $\log |v|$, when v goes to infinity. When $d \neq 2$, expectation and variance are of the same order. Similar results are obtained in the context of p -resistance. The proofs rely on a modified Poincaré inequality due to Falik and Samorodnitsky [7].

1. Introduction

The main goal of this paper is to study the effective resistance between two finite sets of vertices in a random electric network with i.i.d resistances. The infinite grid \mathbb{Z}^d will be the essential graph that we will focus on. Let us first briefly describe our notation for a deterministic electrical network (for more background, see Doyle and Snell [6], Peres [15], Lyons and Peres [12] and Soardi [17]). Let $G = (V, E)$ be an unoriented locally finite graph with an at most countable set of vertices V and a set of edges E (we allow multiple edges between two vertices). Let $r = (r_e)_{e \in E}$ be a collection of positive real numbers, which are called resistances. To each edge e , one may associate two oriented edges, and we shall denote by \vec{E} the set of all these oriented edges. Let A and Z be two finite, disjoint, non-empty sets of vertices of G : A will denote the source of the network, and Z the sink. A function θ on \vec{E} is called a flow from A to Z with strength $\|\theta\|$ if it is antisymmetric, i.e. $\theta_{\vec{xy}} = -\theta_{\vec{yx}}$, if it satisfies the node law at each vertex x of $V \setminus (A \cup Z)$:

$$\sum_{y \sim x} \theta_{\vec{xy}} = 0,$$

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and if the “flow in” at A and the “flow out” at Z equal $\|\theta\|$:

$$\|\theta\| = \sum_{a \in A} \sum_{\substack{y \sim a \\ y \notin A}} \theta(\vec{ay}) = \sum_{z \in Z} \sum_{\substack{y \sim z \\ y \notin Z}} \theta(\vec{yz}).$$

In this definition, it is assumed that all the vertices in A are considered as a single one, as if they were linked by a wire with null resistance (and the same is true for Z). The effective resistance $\mathcal{R}_r(A \leftrightarrow Z)$ may be defined in different ways, the following is the most appropriate for us:

$$\mathcal{R}_r(A \leftrightarrow Z) = \inf_{\|\theta\|=1} \sum_{e \in E} r_e \theta(e)^2, \quad (1)$$

where the infimum is taken over all flows θ from A to Z with strength 1. This infimum is always attained at what is called the *unit minimal (or wired) current* (see [17] Theorem 3.25, p. 40). A current is a flow which satisfies, in addition to the node law, Kirchhoff’s loop law (see [17], p. 12). In finite graphs, currents are unique whereas in infinite graphs, there may exist more than one current. But in \mathbb{Z}^d , for instance, between two finite sets A and Z , it is known to be attained uniquely when the resistances are bounded away from 0 and infinity (see Lyons and Peres [12], p. 82 or Soardi [17], p. 39–43).

Electric networks have been thoroughly studied by probabilists since there is a correspondence between electrical networks on a given graph and reversible Markov chains on the same graph. Let us introduce randomness on the electrical network itself by choosing the resistances independently and identically distributed. That is to say, let ν be a probability measure on \mathbb{R}_+ , and equip \mathbb{R}_+^E with the tensor product $\nu^{\otimes E}$. When the resistances are bounded away from 0 and ∞ , it is easy to see that the mean of the effective resistance is of the same order of that in the network where all resistances are equal to 1. In fact, different realizations of this network are “roughly equivalent” (see Lyons and Peres [12], p. 42), and for example, the associated random walks are of the same type. Related results are those of Berger [5], p. 550 and Pemantle and Peres [14], which give respectively sufficient conditions for almost sure recurrence of the network and a necessary and sufficient condition for almost sure transience. In this paper, we are mainly concerned with the typical fluctuations of the function $r \mapsto \mathcal{R}_r(A \leftrightarrow Z)$ around its mean when A and Z are “far apart”. For simplicity, we choose to focus on the variance of the effective resistance. Typically, we will take A and Z reduced to two vertices far apart: $A = \{a\}$ and $Z = \{z\}$, and we shall note $\mathcal{R}_r(a \leftrightarrow z)$ instead of $\mathcal{R}_r(\{a\} \leftrightarrow \{z\})$. Following the terminology used in First Passage Percolation (see [10]), we shall call this the *point-to-point effective resistance* from a to z .

In this paper, we prove that the type of fluctuations of the point-to-point effective resistance on \mathbb{Z}^d is qualitatively different when $d = 2$ and $d \neq 2$. Indeed, when $d \neq 2$, we will see, quite easily, that these fluctuations are of the same order as its mean. On the other hand, when $G = \mathbb{Z}^2$, and the resistances are bounded away from 0 and ∞ , it is easy to show that the mean of $\mathcal{R}_r(0 \leftrightarrow v)$ is of order $\log |v|$, where $|v|$ stands for the l^1 -norm of the vertex v (see Sect. 3). The main result of this paper is the following variance bound on \mathbb{Z}^2 when the resistances are distributed according to a Bernoulli distribution bounded away from 0.

Theorem 1.1. *Suppose that $\nu = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$, with $0 < a \leq b < +\infty$. Let E be the set of edges in \mathbb{Z}^2 , and define $\mu = \nu^E$. Then, as v goes to infinity:*

$$\text{Var}_\mu(\mathcal{R}_r(0 \leftrightarrow v)) = O\left((\log |v|)^{\frac{2}{3}}\right).$$

Here as in the rest of the article, when f and g are two functions on \mathbb{Z}^d , we use the notation “ $f(v) = O(g(v))$ as v goes to infinity” to mean there is a positive constant C such that, for v large enough,

$$f(v) \leq Cg(v).$$

We shall also use the notation “ $f(v) = \Theta(g(v))$ ” to mean “ $f(v) = O(g(v))$ and $g(v) = O(f(v))$ ”.

The paper is organised as follows. In Sect. 2 we introduce the main tool of this paper, which is a modified Poincaré inequality due to Falik and Samorodnitsky [7]. A first result is given in Proposition 2.2, which announces our main result, on \mathbb{Z}^2 . Section 3 is devoted to the analysis of \mathbb{Z}^2 : we prove our main result, Theorem 1.1 and compare it to the simpler case of \mathbb{Z}^d , for $d \neq 2$. The choice of the Bernoulli setting has been done for the sake of simplicity, but it is possible to extend our variance bound to other distributions, and even to obtain the corresponding exponential concentration inequalities. This is developed in Sect. 4. In Sect. 5, we make some remarks and conjectures on the a priori simpler case of the left-right resistance on the $n \times n$ grid. Finally, Sect. 6 is devoted to an extension of Theorem 1.1 to the non-linear setting of p -networks.

2. A General Result in a Bernoulli Setting

In this section, we suppose that ν is the Bernoulli probability measure $\nu = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$, with $0 < a \leq b < +\infty$ and that for each collection of resistances r in $\Omega = \{a, b\}^E$, there exists a unique current flow between two finite sets of vertices of the graph G . We want to bound from above the variance of the effective resistance. Let us denote:

$$f(r) = \mathcal{R}_r(A \leftrightarrow Z).$$

A first idea is to use the Poincaré inequality, which in this setting is equivalent to the Efron-Stein inequality (see e.g. Steele [19] or Ané et al. [1]):

$$\text{Var}(f) \leq \sum_{e \in E} \|\Delta_e f\|_2^2,$$

where Δ_e is the following discrete gradient:

$$\begin{aligned} \Delta_e f(r) &= \frac{1}{2}(f(r) - f(\sigma_e r)), \\ (\sigma_e r)_{e'} &= \begin{cases} r_{e'} & \text{if } e' \neq e \\ b + a - r_e & \text{if } e' = e \end{cases}. \end{aligned}$$

Let θ_r be a flow attaining the minimum in the definition of $\mathcal{R}_r(a \leftrightarrow z)$. Using the definition of the effective resistance (1),

$$\begin{aligned} f(\sigma_e r) - f(r) &\leq \sum_{e' \in E} (\sigma_e r)_{e'} \theta_r(e')^2 - \sum_{e' \in E} r_{e'} \theta_r(e')^2, \\ f(\sigma_e r) - f(r) &\leq (b - a) \theta_r(e)^2. \end{aligned} \tag{2}$$

For any real number h , we denote by h_+ the number $\max\{h, 0\}$,

$$\begin{aligned} \sum_{e \in E} \|\Delta_e f\|_2^2 &= \frac{1}{2} \sum_{e \in E} \mathbb{E}((f(\sigma_e r) - f(r))_+^2), \\ \sum_{e \in E} \|\Delta_e f\|_2^2 &\leq \frac{(b-a)^2}{2} \mathbb{E}(\sum_{e \in E} \theta_r(e)^4). \end{aligned} \quad (3)$$

It is quite possible that this last bound is sharp in numerous settings of interest, including \mathbb{Z}^2 (see Sect. 3), but in general we do not know how to evaluate the right-hand side of inequality (3). We are just able to bound it from above using the fact that when θ is a unit current flow, $|\theta(e)| \leq 1$ for every edge e . This last fact is intuitive, but for a formal proof, one can see Lyons and Peres [12], p. 49–50. Therefore

$$\begin{aligned} \sum_{e \in E} \|\Delta_e f\|_2^2 &\leq \frac{(b-a)^2}{2a} \mathbb{E}(\sum_{e \in E} r_e \theta_r(e)^2), \\ \sum_{e \in E} \|\Delta_e f\|_2^2 &\leq \frac{(b-a)^2}{2a} \mathbb{E}(f). \end{aligned} \quad (4)$$

We have shown that the variance of f is at most of the order of its mean. It is possible to improve on this, under some suitable assumption, by using the following inequality, due to Falik and Samorodnitsky [7]:

$$\text{Var}(f) \log \frac{\text{Var}(f)}{\sum_{e \in E} \|\Delta_e f\|_1^2} \leq 2 \sum_{e \in E} \|\Delta_e f\|_2^2. \quad (5)$$

In order to state it as a bound on the variance of f , and to avoid repetitions, we present it in a slightly different way:

Lemma 2.1. Falik and Samorodnitsky. *Let f belong to $L^1(\{a, b\}^E)$. Suppose that $\mathcal{E}_1(f)$ and $\mathcal{E}_2(f)$ are two real numbers such that:*

$$\begin{aligned} \mathcal{E}_2(f) &\geq \sum_{e \in E} \|\Delta_e f\|_2^2, \\ \mathcal{E}_1(f) &\geq \sum_{e \in E} \|\Delta_e f\|_1^2, \end{aligned}$$

and

$$\frac{\mathcal{E}_2(f)}{\mathcal{E}_1(f)} \geq e.$$

Then,

$$\text{Var}(f) \leq 2 \frac{\mathcal{E}_2(f)}{\log \frac{\mathcal{E}_2(f)}{\mathcal{E}_1(f) \log \frac{\mathcal{E}_2(f)}{\mathcal{E}_1(f)}}}.$$

Proof. Inequality (5) is proved by Falik and Samorodnitsky only for a finite set E , but it extends straightforwardly to a countable set E , for functions in $L^1(\{a, b\}^E)$. Therefore, we have:

$$\text{Var}(f) \log \frac{\text{Var}(f)}{\mathcal{E}_1(f)} \leq 2\mathcal{E}_2(f). \quad (6)$$

Now, consider the following disjunction:

- either $\text{Var}(f) \leq \frac{\mathcal{E}_2(f)}{\log \frac{\mathcal{E}_2(f)}{\mathcal{E}_1(f)}}$,
- or $\text{Var}(f) \geq \frac{\mathcal{E}_2(f)}{\log \frac{\mathcal{E}_2(f)}{\mathcal{E}_1(f)}}$, and plugging this inequality into (6) gives us:

$$\text{Var}(f) \leq 2 \frac{\mathcal{E}_2(f)}{\log \frac{\mathcal{E}_2(f)}{\mathcal{E}_1(f) \log \frac{\mathcal{E}_2(f)}{\mathcal{E}_1(f)}}}.$$

In any case, since $\mathcal{E}_2(f)/\mathcal{E}_1(f) \geq e$, the second possibility is weaker than the first one, and we get:

$$\text{Var}(f) \leq 2 \frac{\mathcal{E}_2(f)}{\log \frac{\mathcal{E}_2(f)}{\mathcal{E}_1(f) \log \frac{\mathcal{E}_2(f)}{\mathcal{E}_1(f)}}}.$$

□

This inequality is very much in the spirit of an inequality by Talagrand [20] and could be called a modified Poincaré inequality (see also [2, 3] and [16] for more information on such inequalities). The idea to use such a type of inequalities in order to improve variance bounds is due to Benjamini, Kalai and Schramm [4] in the context of First Passage Percolation. In our setting of random electric networks, it allows us to show that as soon as the expected resistance is large but the minimal energy flow via all but few resistances is small, then the variance of the resistance is small compared to the expected resistance. This statement is reflected in the following proposition, which is an introduction to the case of \mathbb{Z}^2 in Sect. 3.

Proposition 2.2. *Let $G = (V, E)$ be an unoriented graph with an at most countable set of vertices V , a set of edges E . Let A and Z be two disjoint non-empty subsets of V . Let a and b be two positive real numbers, and $(r_e)_{e \in E}$ be i.i.d resistances with common law $\frac{1}{2}\delta_a + \frac{1}{2}\delta_b$. Let E_m be any subset of E such that E_m^c is finite. Define:*

$$\alpha_m = \sup_{e \in E_m} \mathbb{E}(r_e \theta_r^2),$$

$$\beta_m = \frac{|E_m^c|}{\mathbb{E}(\mathcal{R}_r(A \leftrightarrow Z))},$$

and

$$\varepsilon_m = \left(\frac{b-a}{a} \right)^2 \alpha_m + (b-a)^2 \beta_m.$$

Suppose that $\varepsilon_m < 1$. Then,

$$\text{Var}(\mathcal{R}_r(A \leftrightarrow Z)) \leq 2K \frac{\mathbb{E}(f)}{\log \frac{K}{\varepsilon_m \log \frac{K}{\varepsilon_m}}},$$

where $K = \sup \left\{ \frac{(b-a)^2}{2a}, e \right\}$.

Proof. We want to use Lemma 2.1. Define:

$$f(r) = \mathcal{R}_r(a \leftrightarrow z).$$

Let us evaluate the terms $\sum_{e \in E} \|\Delta_e f\|_1^2$ and $\sum_{e \in E} \|\Delta_e f\|_2^2$. We have already seen in inequality (4) that:

$$\sum_{e \in E} \|\Delta_e f\|_2^2 \leq \frac{(b-a)^2}{2a} \mathbb{E}(f),$$

and so, define:

$$\mathcal{E}_2(f) = K \mathbb{E}(f),$$

where $K = \sup \left\{ \frac{(b-a)^2}{2a}, e \right\}$. Besides,

$$\begin{aligned} \sum_{e \in E} \|\Delta_e f\|_1^2 &= \sum_{e \in E_m} \|\Delta_e f\|_1^2 + \sum_{e \in E_m^c} \|\Delta_e f\|_1^2, \\ &\leq \frac{b-a}{a} \sum_{e \in E} \|\Delta_e f\|_1 \sup_{e \in E_m} \mathbb{E}(r_e \theta_r^2) + |E_m^c| \sup_{e \in E} \|\Delta_e f\|_1^2, \\ &= \frac{b-a}{a} \alpha_m \sum_{e \in E} \|\Delta_e f\|_1 + \sup_{e \in E} \|\Delta_e f\|_1^2 \beta_m \mathbb{E}(f). \end{aligned}$$

Recall that when θ is a unit current flow, $|\theta(e)| \leq 1$ for every edge e . Therefore,

$$\sup_{e \in E} \|\Delta_e f\|_1 \leq (b-a).$$

Also, using inequality (2),

$$\begin{aligned} \sum_{e \in E} \|\Delta_e f\|_1 &= \sum_{e \in E} \mathbb{E}((f(\sigma_e r) - f(r))_+), \\ &\leq (b-a) \mathbb{E}(\sum_{e \in E} \theta_r(e)^2), \\ &\leq \frac{b-a}{a} \mathbb{E}(f). \end{aligned}$$

Therefore,

$$\sum_{e \in E} \|\Delta_e f\|_1^2 \leq \mathbb{E}(f) \varepsilon_m.$$

Define

$$\mathcal{E}_1(f) = \mathbb{E}(f)\varepsilon_m.$$

The assumption $\varepsilon_m < 1$ ensures that $\mathcal{E}_2(f)/\mathcal{E}_1(f) \geq e$. We conclude by applying Lemma 2.1. \square

Unfortunately, it is not very easy to bound the terms α_m and β_m in an efficient way, essentially because in a random setting, we have no good bound on the amount of current through a particular edge. For example, in Sect. 3, we will have to resort to an averaging trick, and we shall not be able to use directly Proposition 2.2. Nevertheless, for some interesting graphs such as trees, but also the lattices \mathbb{Z}^d , exact calculations are available when all resistances are equal. Therefore, Proposition 2.2 would become more helpful if one could prove the following stability result, which we deliberately state in an informal way.

Question 2.3. *Assume the flow on the fixed resistance 1 environment on a graph satisfies the condition that, except for a small set of edges, only $o(1)$ flow goes via all the other edges, then is the same true for a perturbed environment?*

3. The \mathbb{Z}^d Case

It is natural to inspect the Bernoulli setting on the most studied electrical networks, which are \mathbb{Z}^d , $d \geq 1$. Here, we focus on the point- to-point resistance between the origin and a vertex v when v goes to infinity ($+\infty$ when $d = 1$). Let us denote it as f_v :

$$f_v(r) = \mathcal{R}_r(0 \leftrightarrow v).$$

3.1. The \mathbb{Z}^d case for $d \neq 2$. When $d \neq 2$, one can see easily that the variance of f_v is of the same order as its mean value (when $b > a$). Indeed, when $d = 1$, f_n is just $na + (b - a)B_n$, where B_n is a random variable of binomial distribution with parameters n and $1/2$.

When $d \geq 3$, remark first that, denoting $\bar{a} = (a, a, \dots)$,

$$a\mathcal{R}_{\bar{1}}(0 \leftrightarrow v) \leq \mathcal{R}_r(0 \leftrightarrow v) \leq b\mathcal{R}_{\bar{1}}(0 \leftrightarrow v). \quad (7)$$

Therefore, the mean of f_v is of the same order (up to a multiplicative constant) as in the network where all resistances equal 1. Thus, when $d \geq 3$, the mean of f_v is of order $\Theta(1)$ (see Lyons and Peres [12], p. 39–40). The variance of f_v is also of order $\Theta(1)$ when $b > a$, as follows from the following simple lemma.

Lemma 3.1. *Let $G = (V, E)$ be a unoriented connected graph, $s \in V$ a vertex with finite degree D and Z be a finite subset of V such that $s \notin Z$. If the resistances on E are independently distributed according to a symmetric Bernoulli law on $\{a, b\}$, with $b \geq a > 0$,*

$$\text{Var}(\mathcal{R}_r(s \leftrightarrow Z)) \geq C(b - a)^2,$$

where C is a positive constant depending only on D .

Proof. Denote by $\mathcal{D} = e_1, \dots, e_D$ the D edges incident to s . For any r in \mathbb{R}_+^E , denote

$$f(r) = \mathcal{R}_r(s \leftrightarrow Z),$$

and let $(b^{(D)}, r^{-D})$ be the the set of resistances obtained from r by switching all resistances on \mathcal{D} to b . One has:

$$\begin{aligned} \text{Var}(f) &\geq \mathbb{E}((f - \int f \, dr_{e_1} \dots dr_{e_D})^2), \\ &\geq \frac{1}{2^D} \mathbb{E}((f(b^{(D)}, r^{-D}) - \int f \, dr_{e_1} \dots dr_{e_D})^2), \\ &\geq \left(\frac{1}{2^D}\right)^2 \mathbb{E}((f(b^{(D)}, r^{-D}) - (f(a^{(D)}, r^{-D})))^2), \\ &\geq \left(\frac{1}{2^D}\right)^2 (b-a)^2 \mathbb{E}((\sum_{e \in \mathcal{D}} \theta_{(b^{(D)}, r^{-D})}(e))^2), \\ &\geq \left(\frac{1}{2^D}\right)^2 (b-a)^2 \frac{1}{D^2}. \end{aligned}$$

□

3.2. The case of \mathbb{Z}^2 : some heuristics. Now, let us examine the case of \mathbb{Z}^2 . When the resistances are bounded away from 0 and infinity, f_v , and therefore its expectation, is of order $\Theta(\log |v|)$. Indeed, Eq. (7) implies that it is of the same order as the resistance on the network where all resistances equal 1. This more simple resistance can be explicitly computed using Fourier transform on the lattice \mathbb{Z}^2 (see Soardi [17], p. 104–107). In a more simple way, it can be easily bounded from below by using Nash-Williams inequality, and from above by embedding a suitable tree in \mathbb{Z}^2 (see Doyle and Snell [6], p. 85, or alternatively Lyons and Peres [12], p. 39–40). A more complicated question to address is the existence of a precise limit of the ratio $\mathbb{E}(f_v)/\log |v|$. This would lead to an analog of the “time constant” arising in the context of First Passage Percolation (see Kesten [10]). Closely related questions are the existence of an asymptotic shape and, if it exists, whether it is an euclidean ball or not. We believe that the time constant and the asymptotic shape exist.

Conjecture 3.2. *Define*

$$B_t = \{v \in \mathbb{Z}^2 \text{ s.t. } \mathcal{R}_r(0 \leftrightarrow v) \leq t\}.$$

There exists a non-empty, compact subset of \mathbb{R}^2 , B_0 such that, for every positive number ε ,

$$(1 - \varepsilon)B_0 \subset \frac{1}{\log t} B_t \subset (1 + \varepsilon)B_0.$$

What about the order of the variance of f_v , when v goes to infinity? Reasonably, it should be of order $\Theta(1)$. Since we did not manage to prove this, we state it as a conjecture:

Conjecture 3.3. Suppose that $v = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$, with $0 < a \leq b < +\infty$. Let E be the set of edges in \mathbb{Z}^2 , and define $\mu = v^E$. Then, as v tends to infinity:

$$\text{Var}_\mu(\mathcal{R}_r(0 \leftrightarrow v)) = \Theta(1).$$

A first intuitive support to this conjecture comes from inequality (3). It is quite possible that it gives a bound of order $O(1)$. Indeed, this would be the case if the current in the perturbed environment remained “close” (for example at a l^4 -distance of order 1) to the current in the uniform network (with all resistances equal to 1).

A second support to this conjecture comes from the analysis of the graph $G_n = (V_n, E_n)$. This one arises when one applies in a classical way the Nash-Williams inequality to get a lower bound on the resistance between the origin and the border of the box $\{-n, \dots, n\} \times \{-n, \dots, n\}$. The set of vertices V_n is just $\{0, \dots, n\}$. For i in $\{0, \dots, n-1\}$, draw $2i+1$ parallel edges between i and $i+1$, and call them $e_{i,1}, \dots, e_{i,2i+1}$. This is a Parallel-Series electric network, and the effective resistance is easy to compute:

$$\mathcal{R}_r(0 \leftrightarrow n) = \sum_{i=0}^{n-1} \frac{1}{\sum_{k=1}^{2i+1} \frac{1}{r_{i,k}}}.$$

One can show the following result.

Proposition 3.4. If the resistances on G_n are independently distributed according to a symmetric Bernoulli law on $\{a, b\}$, with $b \geq a > 0$,

$$\mathbb{E}(\mathcal{R}_r(0 \leftrightarrow n)) = \Theta(\log n),$$

and

$$\text{Var}(\mathcal{R}_r(0 \leftrightarrow n)) = \Theta(1).$$

Proof. To shorten the notations, we treat the case $a = 1/2$ and $b = 1$. For any r in $\{a, b\}^E$, denote

$$f(r) = \mathcal{R}_r(0 \leftrightarrow n).$$

The estimate on the mean is obvious. The estimate on the variance is easy too. First note that

$$\text{Var}(f) = \sum_{i=0}^{n-1} \text{Var} \left(\frac{1}{\sum_{k=1}^{2i+1} \frac{1}{r_{i,k}}} \right).$$

Denote by Y_i the random variable:

$$Y_i = \frac{1}{\sum_{k=1}^{2i+1} \frac{1}{r_{i,k}}}.$$

Remark that

$$\sum_{k=1}^{2i+1} \frac{1}{r_{i,k}} = 2i+1 + B_{2i+1},$$

where B_{2i+1} has a binomial distribution with parameters $2i + 1$ and $1/2$. Therefore, denoting

$$N_i = \frac{B_{2i+1} - \frac{2i+1}{2}}{\sqrt{\frac{2i+1}{4}}},$$

when i tends to infinity, N_i converges weakly to a standard Gaussian variable, and

$$Y_i = \frac{1}{3\frac{(2i+1)}{2} + \sqrt{\frac{2i+1}{4}} N_i}.$$

Therefore,

$$Y_i = \frac{1}{9(i + 1/2)^{3/2}} \left[3\sqrt{2i+1} \left(\frac{1}{1 + \frac{N_i}{3\sqrt{2i+1}}} - 1 \right) \right] + \frac{1}{3(i + 1/2)}.$$

Define

$$Z_i = 3\sqrt{2i+1} \left(\frac{1}{1 + \frac{N_i}{3\sqrt{2i+1}}} - 1 \right).$$

Since the sequence (N_i) is weakly convergent, it is bounded in probability. Hence, using that:

$$\frac{1}{1+x} - 1 + x = O(x^2),$$

as x goes to zero, we deduce that $\sqrt{2i+1}(Z_i + N_i)$ is bounded in probability, and therefore Z_i converges in distribution to a standard Gaussian random variable. Using the concentration properties of the binomial distribution, it is easy to show that Z_i and Z_i^2 are asymptotically uniformly integrable. This implies that the variance of Z_i tends to 1 as i tends to infinity. Thus,

$$\text{Var}(Y_i) = \frac{1}{9^2(i + 1/2)^3} \text{Var}(Z_i) = \frac{1}{9(i + 1/2)^3} (1 + o(1)),$$

and consequently,

$$\text{Var}(f) = \Theta(1).$$

Of course, on \mathbb{Z}^2 , things are more difficult to compute. \square

3.3. The case of \mathbb{Z}^2 : proof of Theorem 1.1. We shall prove below that the variance of f_v is of order $O((\log |v|)^{\frac{2}{3}})$. We shall proceed very much as in [4], resorting to an averaging trick to trade the study of f_v against the study of a randomized version of it.

Proof of Theorem 1.1. Let m be a positive integer to be fixed later, and z a random variable, independent from the edge-resistances, and distributed according to μ_m , the uniform distribution on the box $B_m = [0, m-1]^2 \cap \mathbb{Z}^2$. Define

$$\tilde{f}(z, r) := \mathcal{R}_r(z, v+z).$$

We think of \tilde{f} as a function on the space $\tilde{\Omega} := B_m \times \{a, b\}^E$ which is endowed with the probability measure $\mu_m \otimes \nu^{\otimes E}$. The first thing to note is that f_v and \tilde{f} are not too far apart. To see this, we can use the following triangle inequality (see [12], Exercise 2.65, p. 67), which holds for every three vertices x, y, z in \mathbb{Z}^2 , and any $r \in \Omega$,

$$\mathcal{R}_r(x, z) \leq \mathcal{R}_r(x, y) + \mathcal{R}_r(y, z). \quad (8)$$

Therefore, taking L^2 -norms in $L^2(\mu_m \otimes \nu^{\otimes E})$, and noting that $|z| \leq 2m$,

$$\begin{aligned} \|f_v - \tilde{f}\|_2 &\leq \|\mathcal{R}_r(0, z)\|_2 + \|\mathcal{R}_r(z, z+v)\|_2, \\ &\leq C \log m, \end{aligned}$$

where C is a universal constant. Noting that \tilde{f} has the same expectation as f_v , we get thus:

$$\begin{aligned} \|f_v - \mathbb{E}(f_v)\|_2 &\leq \|f_v - \tilde{f}\|_2 + \|\tilde{f} - \mathbb{E}(\tilde{f})\|_2, \\ &\leq C \log m + \|\tilde{f} - \mathbb{E}(\tilde{f})\|_2. \end{aligned}$$

Therefore:

$$\text{Var}(f_v) \leq (C \log m)^2 + 2C \log m \sqrt{\text{Var}(\tilde{f})} + \text{Var}(\tilde{f}). \quad (9)$$

Now, we want to bound the variance of \tilde{f} from above. Define:

$$\begin{aligned} \mathbb{E}_\mu(\tilde{f}) &= \int f(z, r) d\mu_m(z), \\ \mathbb{E}_\nu(\tilde{f}) &= \int f(z, r) d\nu^{\otimes E}(r), \\ \text{Var}_\mu(\tilde{f}) &= \mathbb{E}_\mu(f(z, r)^2) - \mathbb{E}_\mu(f(z, r))^2, \\ \text{Var}_\nu(\tilde{f}) &= \mathbb{E}_\nu(f(z, r)^2) - \mathbb{E}_\nu(f(z, r))^2. \end{aligned}$$

Then, we split the variance of \tilde{f} into two parts: the one due to z and the other due to r ,

$$\text{Var}(\tilde{f}) = \mathbb{E}_\nu(\text{Var}_\mu(\tilde{f})) + \text{Var}_\nu(\mathbb{E}_\mu(\tilde{f})). \quad (10)$$

Thanks to the triangle inequality (8),

$$\mathbb{E}_\nu(\text{Var}_\mu(\tilde{f})) \leq (C \log m)^2. \quad (11)$$

To bound the last term of the sum in (10), we apply Lemma 2.1 to $\mathbb{E}_\mu(\tilde{f})$. Remark that, thanks to Jensen's inequality,

$$\|\Delta_e \mathbb{E}_\mu(\tilde{f})\|_1 \leq \|\Delta_e \tilde{f}\|_1,$$

where the first L^1 -norm integrates against $\nu^{\otimes E}$, and the second one integrates against $\mu_m \otimes \nu^{\otimes E}$. Also,

$$\left\| \Delta_e \mathbb{E}_\mu(\tilde{f}) \right\|_2 \leq \left\| \Delta_e \tilde{f} \right\|_2.$$

Let us denote by θ_r^z the unit current flow from z to $z + v$, when the resistances are r . Using inequality (2), and the translation invariance of this setting, we get, for every edge e :

$$\begin{aligned} \left\| \Delta_e \tilde{f} \right\|_1 &\leq (b-a) \mathbb{E}(\theta_r^z(e)^2), \\ &= (b-a) \mathbb{E}(\theta_r^0(e-z)^2), \\ &= \frac{b-a}{(m+1)^2} \sum_{z_0 \in B_m} \mathbb{E}(\theta_r(e-z_0)^2). \end{aligned}$$

Now we claim that:

$$\forall e \in E, \quad \sum_{z_0 \in B_m} \mathbb{E}(\theta_r(e-z_0)^2) \leq \frac{5b(m+1)}{a}. \quad (12)$$

Assuming this claim, we have:

$$\sup_{e \in E} \left\| \Delta_e \tilde{f} \right\|_1 = O\left(\frac{1}{m}\right). \quad (13)$$

To see that claim (12) is true, let e_- be the left-most or lower end-point of e and remark that the set of edges described by $e - z$ is included in the set of edges of the box $B_m^e := e_- + B_{m+1}$. Therefore,

$$\sum_{z_0 \in B_m} \mathbb{E}(\theta_r(e-z_0)^2) \leq \sum_{e' \subset B_m^e} \mathbb{E}(\theta_r(e')^2).$$

Let ∂B_m^e be the (inner) border of the box B_m^e :

$$\begin{aligned} \partial B_m &= \{0\} \times [0, m] \cup \{m\} \times [0, m] \cup [0, m] \times \{0\} \cup [0, m] \times \{m\}, \\ \partial B_m^e &= e_- + \partial B_m. \end{aligned}$$

First, suppose that neither 0 nor v belongs to B_m^e . We define a flow η from 0 to v such that:

$$\begin{aligned} \eta(e') &= \theta_r(e') \quad \text{if } e' \not\subset B_m^e, \\ \eta(e') &= 0 \quad \text{if } e' \subset B_m^e \text{ and } e' \not\subset \partial B_m^e. \end{aligned}$$

These conditions do not suffice to determine uniquely the flow η , but one can then choose the flow on ∂B_m^e that minimizes the energy $\sum_{e' \in \partial B_m^e} r_{e'} \eta(e')^2$. This is the current flow on ∂B_m^e when the flow entering and going outside ∂B_m^e is fixed by θ_r . For a formal proof

of the existence of such a flow, see Soardi [17] Theorem 2.2, p. 22. This flow has a strength less than 1. Therefore, the flow through each edge of ∂B_m^e is less than 1, and:

$$\begin{aligned}
 \sum_{e' \subset B_m^e} r_e \theta_r(e)^2 &= \sum_{e \in E} r_e \theta_r(e)^2 - \sum_{e \not\subset B_m^e} r_e \theta_r(e)^2, \\
 &\leq \sum_{e \in E} r_e \eta(e)^2 - \sum_{e \not\subset B_m^e} r_e \theta_r(e)^2, \\
 &= \sum_{e \subset \partial B_m^e} r_e \eta(e)^2, \\
 &\leq 4b(m+1).
 \end{aligned}$$

Therefore,

$$\sum_{e' \subset B_m^e} \theta_r(e)^2 \leq \frac{4b(m+1)}{a}.$$

Now, suppose that a or v belongs to B_m^e . Let us say v belongs to B_m^e , the other situation being symmetrical. We define a flow η from 0 to v as before, except that instead of assigning 0 to each value inside B_m^e , we keep a path from ∂B_m^e to v on which the flow is assigned to 1, directed towards v . The same considerations as before lead to

$$\sum_{e' \subset B_m^e} \theta_r(e)^2 \leq \frac{5b(m+1)}{a},$$

and claim (12) is proved.

Finally notice that, using inequality (2):

$$\begin{aligned}
 \sum_{e \in E} \|\Delta_e \tilde{f}\|_1 &\leq \frac{b-a}{a} \mathbb{E}(\tilde{f}), \\
 \sum_{e \in E} \|\Delta_e \tilde{f}\|_2^2 &\leq \frac{(b-a)^2}{2a} \mathbb{E}(\tilde{f}).
 \end{aligned}$$

Recall that

$$\mathbb{E}(\tilde{f}) = \mathbb{E}(f_v) = \Theta(\log |v|).$$

Therefore, there exist constants K and K' such that:

$$\begin{aligned}
 \sum_{e \in E} \|\Delta_e \tilde{f}\|_1^2 &\leq \sum_{e \in E} \|\Delta_e \tilde{f}\|_1 \cdot \sup_{e \in E} \|\Delta_e \tilde{f}\|_1, \\
 &\leq K \frac{\log |v|}{m},
 \end{aligned}$$

and:

$$\sum_{e \in E} \|\Delta_e \tilde{f}\|_2^2 \leq K' \log |v|.$$

Denoting $\mathcal{E}_1(\tilde{f}) = K \frac{\log |v|}{m}$ and $\mathcal{E}_2(\tilde{f}) = K' \log |v|$, the hypotheses of Lemma 2.1 are fulfilled, at least for m larger than $e^{\frac{K}{K'}}$. Assume that m is a function of $|v|$ which goes to infinity when $|v|$ goes to infinity. Lemma 2.1 together with inequalities (10) and (11) gives us:

$$\text{Var}(\tilde{f}) \leq O\left(\frac{\log |v|}{\log m} + (\log m)^2\right).$$

Thus, choose m the greatest integer such that $\log m \leq (\log |v|)^{\frac{1}{3}}$ and the result follows from inequality (9). \square

4. Other Distributions and Exponential Concentration Inequalities

Using a forthcoming paper of Benaim and Rossignol [2], one can derive an exponential concentration inequality on the effective resistance for various distributions, Bernoulli or continuous ones. The only estimates needed to use the results in [2] are the main estimates in the proof of Theorem 1.1, and they can be obtained easily when the resistances are bounded away from 0 and infinity. For example, suppose that v is bounded away from 0 and infinity, and that it is either a Bernoulli distribution or absolutely continuous with respect to the Lebesgue measure with a density which is bounded away from 0 on its support, then, there exist two positive constants C_1 and C_2 such that:

$$\forall t > 0, \mathbb{P}\left(|\mathcal{R}_r(0 \leftrightarrow v) - \mathbb{E}(\mathcal{R}_r(0 \leftrightarrow v))| > t(\log |v|)^{\frac{1}{3}}\right) \leq C_1 e^{-C_2 t}.$$

Whether this result may be extended to distributions which are not bounded away from 0 is still uncertain.

5. Left-Right Resistance on the $n \times n$ -Grid

This section is purely prospective, and focuses on another interesting case of study: the left-right resistance on the $n \times n$ -grid on \mathbb{Z}^2 . The graph is $\mathbb{Z}^2 \cap [0, n] \times [0, n]$, the source is $A_n = \{0\} \times [0, n]$ and the sink is $Z_n = \{n\} \times [0, n]$. When all resistances are equal to 1, one may easily see that $\mathcal{R}_1(A_n \leftrightarrow Z_n)$ equals $n/(n+1)$ and therefore tends to 1, as n tends to infinity. In a random setting, where all resistances are independently and identically distributed with respect to v , this implies that:

$$\limsup_{n \rightarrow \infty} \mathbb{E}(\mathcal{R}_r(A_n \leftrightarrow Z_n)) \leq \int x \, dv(x),$$

where the inequality follows from:

$$\mathbb{E}(\inf_{i \in I} f_i) \leq \inf_{i \in I} \mathbb{E}(f_i).$$

Recall the dual characterisation of the resistance:

$$\frac{1}{\mathcal{R}_r(A_n \leftrightarrow Z_n)} = \inf_F \sum_e \frac{1}{r_e} (F(e_+) - F(e_-))^2,$$

where the infimum is taken over all functions F on the vertices which equal 1 on Z_n and 0 on A_n . This implies:

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left(\frac{1}{\mathcal{R}_r(A_n \leftrightarrow Z_n)} \right) \leq \int \frac{1}{x} d\nu(x).$$

Using $\mathbb{E}(1/R) \geq 1/\mathbb{E}(R)$, we finally obtain:

$$\frac{1}{\int \frac{1}{x} d\nu(x)} \leq \liminf_{n \rightarrow \infty} \mathbb{E}(\mathcal{R}_r(A_n \leftrightarrow Z_n)) \leq \limsup_{n \rightarrow \infty} \mathbb{E}(\mathcal{R}_r(A_n \leftrightarrow Z_n)) \leq \int x d\nu(x).$$

See also Theorem 2 in Hammersley [8]. In fact, it is natural to conjecture that the limit of $\mathcal{R}_r(A_n \leftrightarrow Z_n)$ as n tends to infinity exists almost surely and is constant (see Hammersley [8], p. 350). This is indeed the case, at least under an ellipticity condition, as follows from the work by Künnemann [11] (see Theorem 7.4, p. 230 in the book by Jikov et al. [9]). This work relies on homogenization techniques introduced by Papanicolaou and Varadhan [13]. Notice that in the book by Jikov et al., the law of large numbers is even stated for conductances which are allowed to take the value 0 (see [9] Chapters 8 and 9, notably Eq. (9.16), p. 303 and Theorem 9.6, p. 314).

Returning to resistances with finite mean, the variance in this setting is obviously less than 1, but inequality (3) suggests that it is much lower.

Conjecture 5.1. *If the resistances are bounded away from 0 and infinity,*

$$\text{Var}(\mathcal{R}_r(A_n \leftrightarrow Z_n)) = O\left(\frac{1}{n^2}\right).$$

A lower bound of this order has been proven by Wehr [21] for some absolutely continuous distributions ν under the assumption that the convergence of the effective resistance holds almost surely. Actually, Wehr's result is stated for effective conductivity, that is the inverse of effective resistance, but in this context, they both are of order $\Theta(1)$, and the lower bound of Wehr implies a lower bound of the same order on the variance of the resistance. A very appealing question is therefore:

Question 5.2. *Defining*

$$R_n = \mathcal{R}_r(A_n \leftrightarrow Z_n),$$

does $n(R_n - \mathbb{E}(R_n))$ converge in distribution as n tends to infinity? What is the limit law?

6. Submean Variance Bound for p -Resistance

In the analysis presented so far, the probabilistic interpretation of electrical networks has played no role. It is therefore tempting to extend our work to the setting of p -networks (see for instance Soardi [17], p.176-178). As before, consider an unoriented, at most countable and locally finite graph $G = (V, E)$. Let $r = (r_e)_{e \in E}$ be a collection of resistances. For any $p > 1$, we define the p -resistance between two vertices x and y as

$$\mathcal{R}_r^p(x \leftrightarrow y) = \inf_{\|\theta\|=1} \sum_{e \in E} r_e |\theta(e)|^p, \quad (14)$$

where the infimum is taken over all flows θ from x to y with strength 1. It is known that the p -resistance from 0 to infinity on \mathbb{Z}^d , when all resistances equal 1, is finite if and only if $p > d/(d-1)$ (see Soardi and Yamasaki [18]). More precisely, the flow described in [18] and the usual shorting argument to lowerbound resistance from 0 to the border of the box B_n allow to obtain the following estimate of the p -resistance on \mathbb{Z}^d :

$$\mathbb{E}(\mathcal{R}_r^p(0 \leftrightarrow v)) = \Theta \left(\sum_{k=0}^{|v|} \frac{1}{(2n+1)^{(d-1)(p-1)}} \right).$$

Whenever this expectation tends to infinity as $|v|$ tends to infinity, and when $d \geq 2$, one may hope to obtain a similar result as in Theorem 1.1. This is indeed the case: we obtain a weaker result, but still, the variance is small compared to the mean. The proof is essentially the same as in the case where $p = 2$. There are two main important points to take care of. First, it remains true that for a unit flow which minimizes the p -energy $\sum_{e \in E} r_e |\theta(e)|^p$, the flow on each edge is less than 1. This follows from the same argument as in the linear case (see Lyons and Peres [12], p. 49-50). Second, it is not clear whether inequality (8) remains true or not. Nevertheless, we can easily obtain the following weaker inequality. For every three vertices x, y, z in \mathbb{Z}^d , and any $r \in \Omega$,

$$\mathcal{R}_r^p(x, z) \leq \mathcal{R}_r^p(x, y) + 2^p b |z - y|. \quad (15)$$

To see this, let $\theta^{x,y}$ be the unit current flow (for p -energy) from x to y and $\pi = (u_0 = y, u_1, \dots, u_{|z-y|} = z)$ be a deterministic oriented path from y to z . Define a flow $\theta^{y,z}$ from y to z as follows:

$$\begin{aligned} \theta^{y,z}(e) &= 0, & \text{if } e \notin \pi, \\ \theta^{y,z}(\overrightarrow{u_i u_{i+1}}) &= 1, & \text{if } i \in \{0, \dots, |z-y|-1\}. \end{aligned}$$

Now, let $\eta^{x,z}$ be the unit flow $\theta^{x,y} + \theta^{y,z}$, which goes from x to z .

$$\begin{aligned} \mathcal{R}_r^p(x, z) &\leq \sum_e r_e |\eta^{x,z}(e)|^p, \\ &= \sum_{e \notin \pi} r_e |\theta^{x,y}(e)|^p + \sum_{e \in \pi} r_e |\theta^{x,y}(e) + 1|^p, \\ &\leq \mathcal{R}_r^p(x, y) + 2^p b |z - y|, \end{aligned}$$

where the last inequality follows from the fact that $|\theta_e^{x,y}| \leq 1$ for every edge e . Inequality (15) is proved. This allows to adapt the proof of Theorem 1.1 as follows. Inequalities (9) and (11) become respectively:

$$\text{Var}(f_v) \leq (Cm)^2 + 2Cm \sqrt{\text{Var}(\tilde{f})} + \text{Var}(\tilde{f}),$$

and

$$\mathbb{E}_v(\text{Var}_\mu(\tilde{f})) = O(m^2).$$

The rest of the proof is the same, and leads to:

$$\text{Var}(\tilde{f}) = O \left(\frac{\mathbb{E}(f)}{\log m} + m^2 \right).$$

We can choose, for instance $m = \lceil (\mathbb{E}(f))^{1/3} \rceil$, to get the following weaker analog of Theorem 1.1.

Proposition 6.1. Suppose that $v = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$, with $0 < a \leq b < +\infty$. Let $d \geq 1$ be an integer, E be the set of edges in \mathbb{Z}^d , and define $\mu = v^E$. Then, for any real number p in $]1, +\infty[$:

$$\mathbb{E}(\mathcal{R}_r^p(0 \leftrightarrow v)) = \Theta(a_d(|v|, p)),$$

and if $d \geq 2$,

$$\text{Var}_\mu(\mathcal{R}_r^p(0 \leftrightarrow v)) = O\left(\frac{a_d(|v|, p)}{\log a_d(|v|, p)}\right),$$

where

$$a_d(n, p) = \begin{cases} n^{1-(d-1)(p-1)} & \text{if } p < \frac{d}{d-1}, \\ \log(n) & \text{if } p = \frac{d}{d-1}, \\ 1 & \text{if } p > \frac{d}{d-1}. \end{cases}$$

Remark also that Lemma 3.1 is easily extended to this setting, and we get therefore that, for any $p > \frac{d}{d-1}$,

$$\text{Var}_\mu(\mathcal{R}_r^p(0 \leftrightarrow v)) = \Theta(1).$$

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